Problem 34

$\hat{S}_z|\uparrow\rangle = \frac{1}{2}\hbar|\uparrow\rangle$, i.e. $\hat{\sigma}_z|\uparrow\rangle = |\uparrow\rangle$. Similarly $\hat{\sigma}_z|\downarrow\rangle = -|\downarrow\rangle$

So $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

From $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$, $\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$, $\hat{S}_y = -\frac{i}{2}(\hat{S}_+ - \hat{S}_-)$

$\hat{S}_x|\uparrow\rangle = \frac{1}{2}\hat{S}_+|\uparrow\rangle + \frac{1}{2}\hat{S}_-|\uparrow\rangle = 0 + \frac{1}{2}\hbar\left(\frac{1}{2}\cdot\frac{3}{2} + \frac{1}{2}\cdot\frac{1}{2}\right)^{1/2}|\downarrow\rangle = \frac{1}{2}\hbar|\downarrow\rangle$

$\langle\uparrow|\hat{S}_x|\uparrow\rangle = 0$, $\langle\downarrow|\hat{S}_x|\uparrow\rangle = \frac{1}{2}\hbar$ so $\langle\downarrow|\sigma_x|\uparrow\rangle = 1$.

Similarly $\langle\uparrow|\hat{S}_x|\downarrow\rangle = \frac{1}{2}\hbar$, $\langle\uparrow|\hat{\sigma}_x|\downarrow\rangle = 1$, $\langle\downarrow|\hat{\sigma}_x|\downarrow\rangle = 0$ and

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\hat{S}_y|\uparrow\rangle = -\frac{i}{2}\hat{S}_+|\uparrow\rangle + \frac{i}{2}\hat{S}_-|\uparrow\rangle = +\frac{i}{2}\hbar|\downarrow\rangle$ so $\langle\downarrow|\hat{S}_y|\uparrow\rangle = \frac{i}{2}\hbar$ etc, so

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$