15. For a particle moving in the xy plane, $\hat{H} = \hat{H}_1 + \hat{H}_2$ where $\hat{H}_1 = \frac{\hat{P}_x^2}{2m} + V_1(\hat{x})$ and $\hat{H}_2 = \frac{\hat{P}_y^2}{2m} + V_2(\hat{y})$. $\hat{H}_1|\phi_n\rangle = \varepsilon_n|\phi_n\rangle$ and $\hat{H}_2|\psi_k\rangle = \xi_k|\psi_k\rangle$. Show that $\Psi = \phi_n(x)\psi_k(y)$ are eigenfunctions of $\hat{H}$.

$$\hat{H}\Psi = \left(\frac{\hat{P}_x^2}{2m} + V_1(\hat{x}) + \frac{\hat{P}_y^2}{2m} + V_2(y)\right)\phi_n(x)\psi_k(y) = E\,\phi_n(x)\psi_k(y)$$

$$\psi_k(y)\left(\frac{-\hbar^2}{2m}\right)\frac{\partial^2}{\partial x^2}\phi_n(x) + V_1(\hat{x})\phi_n(x)\psi_k(y) + \phi_n(x)\left(\frac{-\hbar^2}{2m}\right)\frac{\partial^2 \psi_k(y)}{\partial y^2} + V_2(\hat{y})\phi_n(x)\psi_k(y) = E\,\phi_n(x)\psi_k(y)$$

Divide both sides by $\phi_n(x)\psi_k(y)$:

$$\frac{1}{\phi_n(x)}\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_n(x) + V_1(\hat{x}) + \frac{1}{\psi_k(y)}\left(\frac{-\hbar^2}{2m}\right)\frac{\partial^2}{\partial y^2}\psi_k(y) + V_2(\hat{y}) = E = E_x + E_y$$

Since each part only depends on one variable we can write:

$$\frac{1}{\phi_n(x)}\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_n(x) + V_1(\hat{x}) = E_x \quad ; \quad \frac{1}{\psi_k(y)}\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi_k(y) + V_2(\hat{y}) = E_y$$

$$\frac{\hat{P}_x^2}{2m}\phi_n(x) + V_1(\hat{x})\phi_n(x) = E_x\phi_n(x) \quad ; \quad \frac{\hat{P}_y^2}{2m}\psi_k(y) + V_2(\hat{y})\psi_k(y) = E_y\psi_k(y)$$

$$\hat{H}_1\phi_n(x) = E_x\phi_n(x) = \varepsilon_n\phi_n(x) \qquad \hat{H}_2\psi_k(y) = E_y\psi_k(y) = \xi_k\psi_k(y).$$

Therefore $\Psi = \phi_n(x)\psi_k(y)$ is an eigenstate of $\hat{H}$ with eigenvalues of $\varepsilon_n + \xi_k$.