Chem. 540
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PROBLEM FORMALISM 14

Consider a system with exactly two eigenstates (e.g., an electron with spin $\frac{1}{2}$). We will denote the eigenstates of the Hamiltonian $\hat{H}_0$ for this system as $|\phi_1\rangle$ and $|\phi_2\rangle$ and the corresponding eigenvalues ε_1 and ε_2. These states are orthogonal, because the Hamiltonian in a hermitian operator, and we assume that they are also normalized to 1. Thus, the spectral expansion of $\hat{H}_0$ is

$$\hat{H}_0 = \varepsilon_1 |\phi_1\rangle\langle\phi_1| + \varepsilon_2 |\phi_2\rangle\langle\phi_2| .$$

Now suppose that we apply an external field to this system, such that the Hamiltonian is changed to the new Hamiltonian $\hat{H}$, which in terms of the eigenstates of $\hat{H}_0$ has the form

$$\hat{H} = \hat{H}_0 - \hbar\Omega\left(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|\right)$$

where $\Omega > 0$ is a constant.

a) Find the two normalized eigenstates $|\Psi_1\rangle$ and $|\Psi_2\rangle$ of this new Hamiltonian and the corresponding eigenvalues. How many solutions are there for every value of Ω? Is there a possibility for degeneracies?

b) Show explicitly that the eigenstates are orthogonal to one another and that they form a complete set; i.e., they are linearly independent, and any state in this two-dimensional space can be expressed as a linear combination of $|\Psi_1\rangle$ and $|\Psi_2\rangle$.

c) Write down the spectral expansion of $\hat{H}$.

d) What would happen if Ω were a complex number? (Don't try to solve the whole problem again. Just explain what this would mean physically and how the eigenstates and eigenvalues might be qualitatively different.)