Chem. 540
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Problem Formalism 18

To understand an important property of the Dirac δ function consider the integral

$$\int_{-\infty}^{\infty} e^{ikx} dk \ .$$

Clearly the integrand does not go to zero at infinity so the integral is not well behaved. By expanding the exponential into sine and cosine terms, one concludes that this integral must be real valued. Insert a damping factor in the integrand, i.e., consider the following (modified) integral:

$$\int_{-\infty}^{\infty} e^{ikx} e^{-ak^2} dk \ .$$

This new integral is well behaved, so you can evaluate it. Express the result as a function normalized to unity, times the remaining constant factors. Now imagine making the parameter a smaller and smaller, such that in the limit $a \to 0$ we recover the original integral. You will see that the normalized function in your result is more and more sharply peaked about x, while the area under this function remains equal to 1. In the limit $a \to 0$ this function becomes infinitely narrow and thus behaves as a δ function. Therefore establish the following relation:

$$\int_{-\infty}^{\infty} e^{ikx} dk = 2\pi\delta(x) \ .$$