Chem. 540

Formalism Problem 14 - Solution

Def: $$\int_{-\infty}^{\infty} dx\ \delta(x-x_0) f(x) = f(x_0)$$

(a) $$\int_{-\infty}^{\infty} f(x)\,\delta(-x)\,dx \overset{y=-x}{=} \int_{+\infty}^{-\infty} f(-y)\,\delta(y)\,d(-y) = \int_{-\infty}^{\infty} f(-y)\,\delta(y)\,dy$$

$$= f(0) = \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx \quad \Rightarrow \quad \delta(-x) = \delta(x)$$

(b) Substitute $y = ax$. For $a>0$,

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)dx = \int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\delta(y)\,\frac{1}{a}\,dy = \frac{1}{a}f(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} f(x)\delta(x)\,dx.$$

For $a<0$,

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx = \int_{\infty}^{-\infty} f\left(\frac{y}{a}\right)\delta(y)\frac{dy}{a} = -\frac{1}{a}\int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\delta(y)dy = -\frac{1}{a}f(0)$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = \int_{-\infty}^{\infty} f(x)\,\frac{\delta(x)}{|a|}\,dx$$

So $\delta(ax) = \frac{1}{|a|}\delta(x)$.

(c) $$\int_{-\infty}^{\infty} f(x)\,x\,\delta(x)\,dx = f(0)\cdot 0 = 0 \qquad \text{So} \quad x\delta(x) = 0$$

(d) $$\int_{-\infty}^{\infty} f(x)\,x\,\delta'(x)\,dx = f(x)\,\underbrace{x\,\delta(x)}_{0}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}\left(f(x)\cdot x\right)\delta(x)\,dx$$

$$= \int_{-\infty}^{\infty}\left[f'(x)\cdot x + f(x)\right]\delta(x)\,dx$$

$$= \int_{-\infty}^{\infty} f'(x)\,\underbrace{x\,\delta(x)}_{0}\,dx + \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx \quad \Rightarrow \quad x\,\delta'(x) = \delta(x)$$