22. $\hat{H}_0 = \varepsilon_1|\phi_1\rangle\langle\phi_1| + \varepsilon_2|\phi_2\rangle\langle\phi_2|$ $\quad \hat{H} = \hat{H}_0 + c\,(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|)$

a) Find normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ of $\hat{H}$.

$$\begin{pmatrix} \varepsilon_1 & c \\ c & \varepsilon_2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{where } |\psi_1\rangle = \alpha|\phi_1\rangle + \beta|\phi_2\rangle$$

To find E: $\begin{vmatrix} \varepsilon_1 - E & c \\ c & \varepsilon_2 - E \end{vmatrix} = 0 \qquad (\varepsilon_1 - E)(\varepsilon_2 - E) - c^2 = 0$

$$E = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) \pm \tfrac{1}{2}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + 4c^2}$$

$$\left.\begin{aligned} \varepsilon_1\alpha + c\beta &= E\alpha \\ c\alpha + \varepsilon_2\beta &= E\beta \\ \alpha^2 + \beta^2 &= 1 \end{aligned}\right\} \quad \begin{aligned} \beta &= \left(\frac{E-\varepsilon_1}{c}\right)\alpha \\ \alpha &= \frac{c}{\sqrt{c^2 + (E-\varepsilon_1)^2}} \end{aligned}$$

$$|\psi_1\rangle = \frac{c}{\left[c^2 + \left[\frac{1}{2}(\varepsilon_2 - \varepsilon_1) + \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2 + 4c^2}\right]^2\right]^{1/2}}|\phi_1\rangle + \frac{\frac{1}{2}(\varepsilon_2 - \varepsilon_1) + \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+4c^2}}{\left[c^2 + \left[\frac{1}{2}(\varepsilon_2-\varepsilon_1) + \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+4c^2}\right]^2\right]^{1/2}}|\phi_2\rangle$$

with eigenvalue $E_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2 + 4c^2}$

$$|\psi_2\rangle = \frac{\frac{1}{2}(\varepsilon_2 - \varepsilon_1) + \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+4c^2}}{\left[c^2 + \left[\frac{1}{2}(\varepsilon_2-\varepsilon_1) + \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+4c^2}\right]^2\right]^{1/2}}|\phi_1\rangle - \frac{c}{\left[c^2 + \left[\frac{1}{2}(\varepsilon_2-\varepsilon_1) + \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+4c^2}\right]^2\right]^{1/2}}|\phi_2\rangle$$

with eigenvalue $E_2 = \frac{1}{2}(\varepsilon_1+\varepsilon_2) - \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+4c^2}$

There is one solution for each value of c and no possible degeneracies when c is a real valued constant.

b) $$\langle\psi_2|\psi_1\rangle = \frac{c\left(\frac{1}{2}(\varepsilon_2-\varepsilon_1) + \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+4c^2}\right)}{\sqrt{c^2 + (E-\varepsilon_1)^2}}\langle\phi_1|\phi_1\rangle + \frac{-c\left(\frac{1}{2}(\varepsilon_2-\varepsilon_1) + \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+4c^2}\right)}{\sqrt{c^2+(E-\varepsilon_1)^2}}\langle\phi_2|\phi_2\rangle$$

$$\langle\psi_2|\psi_1\rangle = 0$$

Yes they are linearly independent, so any state in the system can be described as a linear combination of $|\psi_1\rangle$ and $|\psi_2\rangle$.

c). $\hat{H} = \left[\frac{1}{2}(\varepsilon_1+\varepsilon_2) + \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+4c^2}\right]|\psi_1\rangle\langle\psi_1| + \left[\frac{1}{2}(\varepsilon_1+\varepsilon_2) - \frac{1}{2}\sqrt{(\varepsilon_1-\varepsilon_2)^2+4c^2}\right]|\psi_2\rangle\langle\psi_2|$.

d). If c were a complex number, the Hamiltonian would no longer be hermitian. You could obtain degenerate or complex eigenvalues.