25. Consider a particle described by the wavefunction:

$$\Psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha}{2}x^2}$$

a) $\langle x \rangle = \int_{-\infty}^{\infty} dx\, \Psi^*(x)\, x\, \Psi(x) = \int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{\pi}}\, x e^{-\alpha x^2} dx$

$\langle x \rangle = \sqrt{\frac{\alpha}{\pi}} \frac{1}{2} \int_{-\infty}^{\infty} e^{-\alpha(u)} du$ where $u = x^2$

$\langle x \rangle = \frac{1}{2}\sqrt{\frac{\alpha}{\pi}}\,(0) = 0$

$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, \Psi^*(x)\, x^2 \Psi(x) = \int_{\infty}^{\infty} dx \sqrt{\frac{\alpha}{\pi}}\, x^2 e^{-\alpha x^2}$

$\langle x^2 \rangle = \sqrt{\frac{\alpha}{\pi}} \left[\frac{x}{-2\alpha} e^{-\alpha x^2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{e^{-\alpha x^2}}{2\alpha} dx \right] = \sqrt{\frac{\alpha}{\pi}} \left[0 + \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \right] = \frac{1}{2\alpha}$

b) $\Psi(p) = (2\pi\hbar)^{-1/2} \int dx\, e^{-ipx/\hbar}\, \Psi(x) = (2\pi\hbar)^{-1/2} \int dx\, e^{-\frac{ipx}{\hbar}} \sqrt[4]{\frac{\alpha}{\pi}}\, e^{-\frac{\alpha}{2}x^2}$

$\Psi(p) = \frac{\sqrt[4]{\alpha}}{\sqrt{4\pi^3\hbar^2}} \int dx\, e^{-\frac{\alpha}{2}(x + ip/\hbar\alpha)^2} e^{-p^2/2\hbar^2\alpha}$

$\Psi(p) = \sqrt[4]{\frac{\alpha}{4\pi^3\hbar^2}} \sqrt[4]{\frac{4\pi^2}{\alpha^2}}\, e^{-p^2/2\hbar^2\alpha} = \left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/4} e^{-p^2/2\hbar^2\alpha}$

c) $\langle p \rangle = \int dp \frac{1}{\sqrt{\pi\alpha\hbar^2}}\, p e^{-p^2/\hbar^2\alpha} = \frac{1}{\sqrt{\pi\alpha\hbar^2}} \frac{1}{2} \int dq\, e^{-\frac{1}{\hbar^2\alpha}(p)} = 0$

$\langle p^2 \rangle = \int dp \frac{1}{\sqrt{\alpha\pi\hbar^2}}\, p^2 e^{-p^2/\hbar^2\alpha}$

$\langle p^2 \rangle = \frac{1}{\sqrt{\alpha\pi\hbar^2}} \left[-\frac{p\hbar^2\alpha}{2} e^{-p^2/\hbar^2\alpha} \Big|_{-\infty}^{\infty} - \int -\frac{\hbar^2\alpha}{2} e^{-p^2/\hbar^2\alpha} dp \right] = \frac{1}{\sqrt{\alpha\pi\hbar^2}} \frac{\sqrt{\pi\alpha\hbar^2}}{1} \left(\frac{\hbar^2\alpha}{2}\right)$

$\langle p^2 \rangle = \hbar^2\alpha/2$

$\langle p \rangle = \int dx\, \Psi^*(x) - i\hbar \frac{\partial}{\partial x} \Psi(x) = \int dx \sqrt{\frac{\alpha}{\pi}}\, i\hbar\alpha x e^{-\alpha x^2}$

$\langle p \rangle = 0$ (as above)

$\langle p^2 \rangle = \int dx\, \Psi^*(x) \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \Psi(x) = \sqrt{\frac{\alpha}{\pi}} \int dx\, e^{-\alpha x^2/2} (\hbar^2\alpha + \hbar^2\alpha^2 x^2) e^{-\alpha x^2/2}$

$\langle p^2 \rangle = \sqrt{\frac{\alpha}{\pi}} \int dx\, \hbar^2\alpha e^{-\alpha x^2} + \sqrt{\frac{\alpha}{\pi}}\, \hbar^2\alpha^2 \int dx\, x^2 e^{-\alpha x^2}$

$\langle p^2 \rangle = 0 + \sqrt{\frac{\alpha}{\pi}}\, \hbar^2\alpha^2 \frac{1}{2\alpha} \sqrt{\frac{\pi}{a}} = \frac{\hbar^2\alpha}{2}$

$\langle p^2 \rangle = \hbar^2\alpha/2.$