8. $[\hat L_x, \hat L_y] = (\hat y\hat p_z - \hat z\hat p_y)(\hat z\hat p_x - \hat x\hat p_z) - (\hat z\hat p_x - \hat x\hat p_z)(\hat y\hat p_z - \hat z\hat p_y) = \hat y\hat p_z\hat z\hat p_x - \cancel{\hat z\hat p_y\hat z\hat p_x} - \cancel{\hat y\hat p_z\hat x\hat p_z} + \hat z\hat p_y\hat x\hat p_z - \hat z\hat p_x\hat y\hat p_z + \cancel{\hat x\hat p_z\hat y\hat p_z} + \cancel{\hat z\hat p_x\hat z\hat p_y} - \hat x\hat p_z\hat z\hat p_y$

$$[\hat L_x, \hat L_y] = \hat y\frac{\hbar}{i}\frac{\partial}{\partial z}\left(\hat z\frac{\hbar}{i}\frac{\partial}{\partial x}\right) + \hat z\frac{\hbar}{i}\frac{\partial}{\partial y}\left(x\frac{\hbar}{i}\frac{\partial}{\partial z}\right) - z\frac{\hbar}{i}\frac{\partial}{\partial x}\left(y\frac{\hbar}{i}\frac{\partial}{\partial z}\right) - x\frac{\hbar}{i}\frac{\partial}{\partial z}\left(\hat z\frac{\hbar}{i}\frac{\partial}{\partial y}\right)$$

$$= \hat y\frac{\hbar}{i}\left(\frac{\hbar}{i}\frac{\partial}{\partial x} + \cancel{\hat z\frac{\hbar}{i}\frac{\partial^2}{\partial z\partial x}}\right) + \cancel{\hat z\hat x\frac{\hbar^2}{i^2}\frac{\partial^2}{\partial y\partial z}} - \cancel{\hat z\hat y\frac{\hbar^2}{i^2}\frac{\partial^2}{\partial x\partial z}} - x\frac{\hbar^2}{i^2}\frac{\partial}{\partial y} - \cancel{xz\frac{\hbar^2}{i^2}\frac{\partial^2}{\partial z\partial y}}$$

$$= \frac{\hbar}{i}\hat y\hat p_x - \frac{\hbar}{i}\hat x\hat p_y = i\hbar(\hat x\hat p_y - \hat y\hat p_x) = i\hbar\hat L_z$$

$$[\hat L_y, \hat L_z] = (\hat z\hat p_x - \hat x\hat p_z)(\hat x\hat p_y - \hat y\hat p_x) - (\hat x p_y - \hat y\hat p_x)(\hat z\hat p_x - \hat x\hat p_z)$$

$$= \hat z\hat p_x\hat x\hat p_y - \cancel{\hat z\hat p_x\hat y\hat p_x} - \cancel{\hat x\hat p_z\hat x\hat p_y} + \hat x\hat p_z\hat y\hat p_x - \hat x\hat p_y\hat z\hat p_x + \cancel{\hat x\hat p_y\hat x\hat p_z} + \cancel{\hat y\hat p_x\hat z\hat p_x} - \hat y\hat p_x\hat x\hat p_z$$

$$= \hat z\left(-i\hbar\frac{\partial}{\partial x}\right)\hat x\left(-i\hbar\frac{\partial}{\partial y}\right) + \hat x\left(-i\hbar\frac{\partial}{\partial z}\right)\hat y\left(-i\hbar\frac{\partial}{\partial x}\right) - \hat x\left(-i\hbar\frac{\partial}{\partial y}\right)\hat z\left(-i\hbar\frac{\partial}{\partial x}\right) - \hat y\left(-i\hbar\frac{\partial}{\partial x}\right)\hat x\left(-i\hbar\frac{\partial}{\partial z}\right)$$

$$= -\hbar^2\hat z\frac{\partial}{\partial y} - \cancel{\hbar^2\hat x\hat z\frac{\partial^2}{\partial x\partial y}} - \cancel{\hbar^2\hat x\hat y\frac{\partial^2}{\partial z\partial x}} + \cancel{\hbar^2\hat x\hat z\frac{\partial^2}{\partial y\partial x}} + \hbar^2\hat y\frac{\partial}{\partial z} + \cancel{\hbar^2\hat y\hat x\frac{\partial^2}{\partial x\partial z}}$$

$$= -\hbar^2\hat z\frac{\partial}{\partial y} + \hbar^2\hat y\frac{\partial}{\partial z} = i\hbar(-\hat z\hat p_y + \hat y\hat p_z) = i\hbar\hat L_x$$

$$[\hat L_z, \hat L_x] = (\hat x\hat p_y - \hat y\hat p_x)(\hat y\hat p_z - \hat z\hat p_y) - (\hat y\hat p_z - \hat z\hat p_y)(\hat x\hat p_y - \hat y\hat p_x)$$

$$= \hat x\hat p_y\hat y\hat p_z - \cancel{\hat x\hat p_y\hat z\hat p_y} - \cancel{\hat y\hat p_x\hat y\hat p_z} + \hat y\hat p_x\hat z\hat p_y - \hat y\hat p_z\hat x\hat p_y + \cancel{\hat z\hat p_y\hat x\hat p_y} + \cancel{\hat y\hat p_z\hat y\hat p_x} - \hat z\hat p_y\hat y\hat p_x$$

$$= \hat x\left(-i\hbar\frac{\partial}{\partial y}\right)\hat y\left(-i\hbar\frac{\partial}{\partial z}\right) - \hat y\left(-i\hbar\frac{\partial}{\partial x}\right)\hat z\left(-i\hbar\frac{\partial}{\partial y}\right) - \hat y\left(-i\hbar\frac{\partial}{\partial z}\right)\hat x\left(-i\hbar\frac{\partial}{\partial y}\right) - \hat z\left(-i\hbar\frac{\partial}{\partial y}\right)\hat y\left(-i\hbar\frac{\partial}{\partial x}\right)$$

$$= (-i\hbar)^2 x\frac{\partial}{\partial z} + \cancel{(-i\hbar)^2 xy\frac{\partial^2}{\partial y\partial z}} + \cancel{(-i\hbar)^2 yz\frac{\partial^2}{\partial x\partial y}} - (-i\hbar)^2 yx\frac{\partial^2}{\partial z\partial y} - (-i\hbar)^2 z\frac{\partial}{\partial x} - \cancel{(-i\hbar)^2 zy\frac{\partial^2}{\partial y\partial x}}$$

$$= -i\hbar x p_z + i\hbar z p_x = i\hbar(z p_x - x p_z) = i\hbar\hat L_y$$

$$[\hat L^2, \hat L_z] = [\hat L_x^2 + \hat L_y^2 + \hat L_z^2, \hat L_z] = [\hat L_x^2, \hat L_z] + [\hat L_y^2, \hat L_z] + [\hat L_z^2, \hat L_z]$$

Rules for commutators: $[AB, C] = [A, C]B + A[B, C]$

$$[\hat L^2, \hat L_z] = [\hat L_x, \hat L_z]\hat L_x + \hat L_x[\hat L_x, \hat L_z] + [\hat L_y, \hat L_z]\hat L_y + \hat L_y[\hat L_y, \hat L_z] + [\hat L_z, \hat L_z]\hat L_z + \hat L_z[\hat L_z, \hat L_z]$$

$$= -i\hbar\hat L_y\hat L_x + -i\hbar\hat L_x\hat L_y + +i\hbar\hat L_x\hat L_y + i\hbar\hat L_y\hat L_x + 0\hat L_z + \hat L_z(0)$$

$$= 0$$