9. $|\varphi_1\rangle$ an $|\varphi_2\rangle$ are two non-orthogonal eigenstates (normalized) of $\hat{A}$, with $\langle\varphi_1|\varphi_2\rangle = \lambda$.

a) Construct orthogonal states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ that have the same eigenvalue.
Let $|\Phi_1\rangle = |\varphi_1\rangle$ and $|\Phi_2\rangle = a|\varphi_1\rangle + b|\varphi_2\rangle$.
Two conditions must be true:
$|\Phi_2\rangle$ must be normalized ($\langle\Phi_2|\Phi_2\rangle = 1$) [Note: $|\Phi_1\rangle$ is already normalized)
$|\Phi_1\rangle$ and $|\Phi_2\rangle$ must be orthoganal: $\langle\Phi_1|\Phi_2\rangle = 0$.

$\langle\Phi_2|\Phi_2\rangle = a^2\langle\phi_1|\phi_1\rangle + b^2\langle\varphi_2|\varphi_2\rangle + ab\langle\varphi_1|\varphi_2\rangle + ab\langle\varphi_2|\varphi_1\rangle = 0$ (Assuming $a, b, \lambda =$ real)
$= a^2 + b^2 + 2ab\lambda = 1$

$\langle\Phi_1|\Phi_2\rangle = a\langle\phi_1|\phi_1\rangle + b\langle\phi_1|\phi_2\rangle = 0$
$= a + b\lambda = 0$.

Solving these equations simultaneously gives:
$$a = \frac{-\lambda}{\sqrt{1-\lambda^2}} \qquad b = \frac{1}{\sqrt{1-\lambda^2}}$$
Therefore: $|\Phi_1\rangle = |\varphi_1\rangle$ and $|\Phi_2\rangle = \frac{-\lambda}{\sqrt{1-\lambda^2}}|\varphi_1\rangle + \frac{1}{\sqrt{1-\lambda^2}}|\varphi_2\rangle$.

$\hat{A}|\Phi_1\rangle = \hat{A}|\varphi_1\rangle = A_1|\varphi_1\rangle \quad \hat{A}|\Phi_2\rangle = \frac{-\lambda}{\sqrt{1-\lambda^2}}\hat{A}|\varphi_1\rangle + \frac{1}{\sqrt{1-\lambda^2}}\hat{A}|\varphi_2\rangle$
$= \frac{-\lambda}{\sqrt{1-\lambda^2}}A_1|\varphi_1\rangle + \frac{1}{\sqrt{1-\lambda^2}}A_2|\varphi_2\rangle$

$\Rightarrow$ Since $A_1 = A_2 = A_{Deg}$: $\hat{A}|\Phi_2\rangle = A_{Deg}\left(\frac{-\lambda}{\sqrt{1-\lambda^2}}|\varphi_1\rangle + \frac{1}{\sqrt{1-\lambda^2}}|\varphi_2\rangle\right) = A_{Deg}|\Phi_2\rangle$
$\hat{A}|\Phi_1\rangle = A_{Deg}(|\varphi_1\rangle) = A_{Deg}|\Phi_2\rangle$.

Yes, $|\Phi_2\rangle$ and $|\Phi_1\rangle$ retain the same, degenerate eigenvalue.

b) If $|\varphi_1\rangle$ and $|\varphi_2\rangle$ have eigenvalue α, and other states $|\Phi_n\rangle$ ($n>2$) have eigenvalues α_n, are $|\varphi_1\rangle$ and $|\varphi_2\rangle$ orthogonal to $|\Phi_n\rangle$?
Assuming $\hat{A}$ is a hermitian operator;
$\langle\varphi_1|\hat{A}|\Phi_n\rangle = \langle A\varphi_1|\Phi_n\rangle$
$\alpha_n\langle\varphi_1|\Phi_n\rangle = \alpha\langle\varphi_1|\Phi_n\rangle \Rightarrow (\alpha_n - \alpha)\langle\varphi_1|\Phi_n\rangle = 0$.
Since $\alpha_n \neq \alpha$, $\langle\varphi_1|\Phi_n\rangle$ must equal zero, therefore $|\varphi_1\rangle$ and $|\Phi_n\rangle$ are orthogonal.
Since $|\Phi_1\rangle, |\Phi_2\rangle$ and $|\varphi_2\rangle$ all have eigenvalue α, the same logic applies.

c) States $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are not uniquely defined. $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are two orthogonal states within a plane and therefore any rotation would preserve their eigenvalues and orthonormality