10. We know that $\hat{A}|a\rangle = \lambda|a\rangle$ and $f(\hat{A}) = e^{\hat{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\hat{A}^k$.

Show that $|a\rangle$ is an eigenstate of $f(\hat{A})$ and find its eigenvalue.

$f(\hat{A})|a\rangle = e^{\hat{A}}|a\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}\hat{A}^k|a\rangle$.

Is $\hat{A}^k|a\rangle = c_k|a\rangle$?

For $k=3$: $\hat{A}^3|a\rangle = \hat{A}\hat{A}\hat{A}|a\rangle = \hat{A}\hat{A}(\lambda|a\rangle) = \lambda\hat{A}(\lambda|a\rangle) = \lambda^3|a\rangle$.

Yes: $\hat{A}^k|a\rangle = c_k|a\rangle$, where $c_k = \lambda^k$.

$e^{\hat{A}}|a\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}\hat{A}^k|a\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k|a\rangle$.

$|a\rangle$ is an eigenstate of $e^{\hat{A}}$ with an eigenvalue of $\sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k$.