11. Show that $e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}}$ iff $[\hat{A},\hat{B}]=0$.

Using the relation in Problem 10:

$$e^{\hat{A}+\hat{B}} = \sum_{k=0}^{\infty} \frac{1}{k!}(\hat{A}+\hat{B})^k$$

$$= 1 + (\hat{A}+\hat{B})^1 + \frac{1}{2}(\hat{A}^2+\hat{B}^2+\hat{B}\hat{A}+\hat{A}\hat{B}) + \frac{1}{6}(\hat{A}^3+\hat{A}\hat{B}^2+\hat{A}\hat{B}\hat{A}+\hat{A}^2\hat{B}+\hat{B}\hat{A}^2 + \hat{B}^3+\hat{B}^2\hat{A}+\hat{B}\hat{A}\hat{B}) + \ldots$$

$$e^{\hat{A}} e^{\hat{B}} = \left(\sum_{k=0}^{\infty}\frac{1}{k!}\hat{A}^k\right)\left(\sum_{m=0}^{\infty}\frac{1}{m!}\hat{B}^m\right) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{k!\,m!}\hat{A}^k\hat{B}^m$$

$$= \left(1+\hat{A}+\frac{1}{2}\hat{A}^2+\frac{1}{6}\hat{A}^3+\ldots\right)\left(1+\hat{B}+\frac{1}{2}\hat{B}^2+\frac{1}{6}\hat{B}^3+\ldots\right)$$

$$= 1 + (\hat{A}+\hat{B}) + \left(\hat{A}\hat{B}+\frac{1}{2}\hat{B}^2+\frac{1}{2}\hat{A}^2\right) + \left(\frac{1}{2}\hat{A}\hat{B}^2+\frac{1}{2}\hat{A}^2\hat{B}+\frac{1}{6}\hat{B}^3+\frac{1}{6}\hat{A}^3\right)$$

+ terms of higher order.

$e^{\hat{A}+\hat{B}}$ would equal $e^{\hat{A}} e^{\hat{B}}$ if:

$$\left.\begin{aligned} \frac{1}{2}(\hat{B}\hat{A}+\hat{A}\hat{B}) &= \hat{A}\hat{B} \\ \frac{1}{6}(\hat{A}\hat{B}^2+\hat{B}^2\hat{A}+\hat{B}\hat{A}\hat{B}) &= \frac{1}{2}\hat{A}\hat{B}^2 \\ \frac{1}{6}(\hat{A}\hat{B}\hat{A}+\hat{A}^2\hat{B}+\hat{B}\hat{A}^2) &= \frac{1}{2}\hat{A}^2\hat{B} \end{aligned}\right\}$$

If $[\hat{A},\hat{B}]=0$, then $\hat{A}\hat{B}=\hat{B}\hat{A}$

and $\hat{A}\hat{B}^2=\hat{B}^2\hat{A}=\hat{B}\hat{A}\hat{B}$

and $\hat{A}^2\hat{B}=\hat{A}\hat{B}\hat{A}=\hat{B}\hat{A}^2$.

Therefore, if $[\hat{A},\hat{B}]=0$ $\quad e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}}$.