

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger)$$

$$\begin{aligned} \langle i | \hat{x} | j \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \langle i | \hat{a} + \hat{a}^\dagger | j \rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{j} \langle i | j-1 \rangle + \sqrt{j+1} \langle i | j+1 \rangle) \\ &= \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{j} \delta_{i,j-1} + \sqrt{j+1} \delta_{i,j+1}) \end{aligned}$$

$$\begin{aligned} \langle \hat{x}^2 \rangle &= \langle \phi_n | \hat{x}^2 | \phi_n \rangle = \sum_m \langle \phi_n | \hat{x} | \phi_m \rangle \langle \phi_m | \hat{x} | \phi_n \rangle \\ &= \frac{\hbar}{2m\omega} \sum_m (\sqrt{m} \delta_{n,m-1} + \sqrt{m+1} \delta_{n,m+1})^2 \\ &= \frac{\hbar}{2m\omega} \left[(\underbrace{\sqrt{n+1}}_{\uparrow m=n+1} + 0)^2 + (0 + \underbrace{\sqrt{n}}_{\uparrow m=n-1})^2 \right] = \frac{\hbar}{2m\omega} (2n+1) \end{aligned}$$