28. Calculate the expectation values of $\hat{x}$, $\hat{x}^2$, $\hat{p}$, and $\hat{p}^2$ for the $n^{th}$ eigenstate of a harmonic oscillator.

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\,\hat{p} \qquad \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\,\hat{p}$$

$$\hat{a} + \hat{a}^+ = 2\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \implies \hat{x} = \sqrt{\frac{2\hbar}{m\omega}}\,\frac{1}{2}(\hat{a} + \hat{a}^+)$$

$$\hat{a} - \hat{a}^+ = \frac{2i}{\sqrt{2m\omega\hbar}}\,\hat{p} \implies \hat{p} = \frac{\sqrt{2m\omega\hbar}}{2i}(\hat{a} - \hat{a}^+)$$

$$\langle n|\hat{x}|n\rangle = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^+|n\rangle\right) = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\sqrt{n}\,\langle n|n-1\rangle + \sqrt{n+1}\,\langle n|n+1\rangle\right)$$

$$\langle n|\hat{x}|n\rangle = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\sqrt{n}\,(0) + \sqrt{n+1}\,(0)\right) = 0$$

$$\langle n|\hat{p}|n\rangle = \frac{\sqrt{2m\omega\hbar}}{2i}\left(\langle n|\hat{a}|n\rangle - \langle n|\hat{a}^+|n\rangle\right) = \frac{\sqrt{2m\omega\hbar}}{2i}\left(\sqrt{n}\,\langle n|n-1\rangle - \sqrt{n+1}\,\langle n|n+1\rangle\right)$$

$$\langle n|\hat{p}|n\rangle = 0$$

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\left(\langle n|\hat{a}\hat{a}|n\rangle + \langle n|a^+a|n\rangle + \langle n|aa^+|n\rangle + \langle n|a^+a^+|n\rangle\right)$$

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{n(n-1)}\,\langle n|n-2\rangle + \sqrt{(n)(n)}\,\langle n|n\rangle + \sqrt{(n+1)(n+1)}\,\langle n|n\rangle + \sqrt{(n+1)(n+2)}\,\langle n|n+2\rangle\right)$$

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}(n + n + 1) = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)$$

$$\langle n|\hat{p}^2|n\rangle = -\frac{m\omega\hbar}{2}\left(\langle n|\hat{a}\hat{a}|n\rangle - \langle n|\hat{a}^+\hat{a}|n\rangle - \langle n|\hat{a}\hat{a}^+|n\rangle + \langle n|\hat{a}^+\hat{a}^+|n\rangle\right)$$

$$\langle n|\hat{p}^2|n\rangle = -\frac{m\omega\hbar}{2}\left(\sqrt{n(n-1)}\,\langle n|n-2\rangle - \sqrt{(n)(n)}\,\langle n|n\rangle - \sqrt{(n+1)(n+1)}\,\langle n|n\rangle + \sqrt{(n+1)(n+2)}\,\langle n|n+2\rangle\right)$$

$$\langle n|\hat{p}^2|n\rangle = -\frac{m\omega\hbar}{2}(-n - n - 1) = m\omega\hbar\left(n + \frac{1}{2}\right)$$