(a) $\langle 0|\hat{x}^2|2\rangle = \dfrac{\hbar}{2m\omega}\left(\langle 0|\hat{a}\hat{a}|2\rangle + \langle 0|\hat{a}^{+}\hat{a}|2\rangle + \langle 0|\hat{a}\hat{a}^{+}|2\rangle + \langle 0|\hat{a}^{+}\hat{a}^{+}|2\rangle\right)$

$= \dfrac{\hbar}{2m\omega}\left(\sqrt{2}\sqrt{1}\,\langle 0|0\rangle + \sqrt{2}\sqrt{2}\,\langle 0|2\rangle + \sqrt{3}\sqrt{3}\,\langle 0|2\rangle + \sqrt{3}\sqrt{4}\,\langle 0|4\rangle\right)$

$= \dfrac{1}{\sqrt{2}}\,\dfrac{\hbar}{m\omega}$

(b) $\langle 0|\hat{x}^2|1\rangle = \dfrac{\hbar}{2m\omega}\left(\cancel{\langle 0|\hat{a}\hat{a}|1\rangle}^{\,0} + \cancel{\langle 0|\hat{a}^{+}\hat{a}|1\rangle}^{\,0} + \cancel{\langle 0|\hat{a}\hat{a}^{+}|1\rangle}^{\,0} + \cancel{\langle 0|\hat{a}^{+}\hat{a}^{+}|1\rangle}^{\,0}\right)$

$\propto |0\rangle,$

$\langle 0|\hat{a}|0\rangle = 0$

$= 0$

(Could have guessed based on symmetry.)

(c) $\langle 1|\hat{x}^3|4\rangle = \left(\dfrac{\hbar}{2m\omega}\right)^{3/2}\left(\langle 1|\hat{a}\hat{a}\hat{a}|4\rangle + \cdots \text{ (all give zero)}\right)$

$= \left(\dfrac{\hbar}{2m\omega}\right)^{3/2}\left(\sqrt{4}\sqrt{3}\sqrt{2}\,\langle 1|1\rangle\right) = \sqrt{3}\left(\dfrac{\hbar}{m\omega}\right)^{3/2}.$

(d) $\langle n|\hat{x}^2|n-1\rangle = \int dx\, \phi_n(x)\, x^2\, \phi_{n-1}(x) = 0$ by symmetry,

since the product $\phi_n \phi_{n-1}$ is an odd function

(e) $\langle n|\hat{x}^2|n+2\rangle = \dfrac{\hbar}{2m\omega}\left(\langle n|\hat{a}\hat{a}|n+2\rangle + \cdots \text{ (all zero)}\right)$

$= \dfrac{\hbar}{2m\omega}\sqrt{n+2}\,\sqrt{n+1}$