Example: perturbation of an oscillator

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

$$\hat{V} = \frac{1}{2} b\hat{x}^2$$

Need matrix elements of x^2:

$$\langle \phi_n | \hat{x}^2 | \phi_{n'} \rangle = \left(\frac{\hbar}{2m\omega}\right) \Big\{ (2n+1)\,\delta_{n,n'} + \sqrt{(n+1)(n+2)}\;\delta_{n',n+2}$$

$$+ \sqrt{n(n-1)}\;\delta_{n',n-2} \Big\}$$

1st order perturbation theory:

$$E_n^{(1)} = \langle \phi_n | \hat{V} | \phi_n \rangle = \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) \cdot \frac{1}{2} b$$

2nd order:

$$E_n^{(2)} = -\sum_{n' \neq n} \frac{|\langle \phi_n | \hat{V} | \phi_{n'} \rangle|^2}{\varepsilon_{n'} - \varepsilon_n}$$

$$= -\left(\frac{1}{2} b\right)^2 \left(\frac{\hbar}{2m\omega}\right)^2 \left\{ \frac{n(n-1)}{-2\hbar\omega} + \frac{(n+1)(n+2)}{2\hbar\omega} \right\}$$

$$= -\hbar\omega\left(n+\frac{1}{2}\right) \cdot \frac{b^2}{8(m\omega^2)^2}$$

So the energy through 2nd order is

$$E_n \simeq \left(n+\frac{1}{2}\right)\hbar\omega \left\{ 1 + \frac{b}{2m\omega^2} - \frac{b^2}{8(m\omega^2)^2} \right\}$$

The exact solution is

$$E_n = \left(n + \frac{1}{2}\right) \hbar \sqrt{\frac{m\omega^2 + b}{m}}$$

$$= \left(n + \frac{1}{2}\right) \hbar\omega \left(1 + \frac{b}{m\omega^2}\right)^{1/2}$$

$$= \left(n + \frac{1}{2}\right) \hbar\omega \left(1 + \frac{b}{2m\omega^2} - \frac{1}{8}\left(\frac{b}{m\omega^2}\right)^2 + \cdots\right) \checkmark$$