Variational principle – Quadratic and quartic potentials

$$\psi_\lambda(x) = e^{-\lambda x^2}$$

$$\int_{-\infty}^{\infty} \psi_\lambda(x)^2 dx = \int_{-\infty}^{\infty} e^{-2\lambda x^2} dx = \sqrt{\frac{\pi}{2\lambda}}$$

$$\int_{-\infty}^{\infty} \psi_\lambda(x) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_\lambda(x)\right) dx = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} (-2\lambda + 4\lambda^2 x^2)\, e^{-2\lambda x^2} dx$$

$$= \sqrt{\frac{\pi}{2\lambda}}\ \frac{\hbar^2 \lambda}{2m}$$

$$\int_{-\infty}^{\infty} \psi_\lambda(x)\ \frac{1}{2}m\omega^2 x^2\ \psi_\lambda(x)\, dx = \sqrt{\frac{\pi}{2\lambda}}\ \frac{m\omega^2}{8\lambda}$$

$$\int_{-\infty}^{\infty} \psi_\lambda(x)\ bx^4\ \psi_\lambda(x)\, dx = \frac{3}{4}\sqrt{\pi}\,(2\lambda)^{-5/2} = \sqrt{\frac{\pi}{2\lambda}}\cdot\frac{3}{4}\frac{1}{4\lambda^2}$$

For quadratic potential,

$$E_\lambda = \frac{\hbar^2\lambda}{2m} + \frac{m\omega^2}{8\lambda} \qquad \frac{\partial E_\lambda}{\partial\lambda} = \frac{\hbar^2}{2m} - \frac{m\omega^2}{8\lambda^2} \qquad \lambda = \frac{m\omega}{2\hbar}$$

For this λ we recover the exact eigenfunction and eigenvalue, $E = \frac{1}{2}\hbar\omega$.

For quartic potential,

$$E_\lambda = \frac{\hbar^2\lambda}{2m} + \frac{3}{16\lambda^2} \qquad \frac{\partial E_\lambda}{\partial\lambda} = \frac{\hbar^2}{2m} - \frac{3}{8\lambda^3} \qquad \lambda^3 = \frac{3m}{4\hbar^2}$$

$$\lambda = \left(\frac{3m}{4\hbar^2}\right)^{1/3}$$

So our approximation to the ground state energy is

$$E = \frac{\hbar^2}{2m}\left(\frac{3m}{4\hbar^2}\right)^{1/3} + \frac{3}{16}\left(\frac{4\hbar^2}{3m}\right)^{2/3} = \left[\frac{1}{2}\left(\frac{3}{4}\right)^{1/3} + \frac{3}{16}\left(\frac{4}{3}\right)^{2/3}\right]\frac{\hbar^{4/3}}{m^{2/3}}$$