40. Trial function: $|\Psi\rangle$.
Ground State: $|\Phi_0\rangle$ where $\langle\Psi|\Phi_0\rangle = 0$.

$$\Psi(x) \equiv \sum_{m\neq 0} C_m \Phi_m = \sum_{m\geq 1} C_m \phi_m \quad \text{since } C_0 = \langle\Psi|\Phi_0\rangle = 0.$$

$$\frac{\langle\Psi|\hat{H}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\sum_n\sum_{n'} C_n^* C_{n'} \langle\Phi_n|\hat{H}|\Phi_{n'}\rangle}{\sum_n\sum_{n'} C_n^* C_{n'} \langle\Phi_n|\Phi_{n'}\rangle} = \frac{\sum_n |C_n|^2 E_n}{\sum_n |C_n|^2}$$

Since $C_0 = 0$:

$$= \frac{\sum_{n>0} |C_n|^2 E_n}{\sum_{n>0} |C_n|^2} \geq \frac{\sum_{n\geq 0} |C_n|^2 E_1}{\sum_{n>0} |C_n|^2}$$

~~Since~~ $\geq E_1 \frac{\sum |C_n|^2}{\sum_{n>0} |C_n|^2}$. Therefore: $\frac{\langle\Psi|\hat{H}|\Psi\rangle}{\langle\Psi|\Psi\rangle} \geq E_1$.

For harmonic oscillator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$ & $\Psi(x) = c\,x\,e^{-\lambda x^2}$.

$$\langle\Psi|\hat{H}|\Psi\rangle = -\frac{\hbar^2}{2m}\langle\Psi|\frac{\partial^2}{\partial x^2}|\Psi\rangle + \frac{1}{2}m\omega^2\langle\Psi|\hat{x}^2|\Psi\rangle$$

$$= \frac{-\hbar^2}{2m}c^2\int dx\,(x e^{-\lambda x^2})(-6\lambda x + 4\lambda^2 x^3)e^{-\lambda x^2} + \frac{1}{2}m\omega^2 c^2\int dx\, x^4 e^{-2\lambda x^2}$$

$$\langle\Psi|\hat{H}|\Psi\rangle = -\frac{\hbar^2 c^2}{2m}\sqrt{\frac{\pi}{2\lambda}}\left(\frac{-3}{4}\right) + \frac{3m\omega^2 c^2}{32\lambda^2}\sqrt{\frac{\pi}{2\lambda}}$$

$$\langle\Psi|\Psi\rangle = \frac{c^2}{4\lambda}\sqrt{\frac{\pi}{2\lambda}}$$

$$E_\lambda = \frac{\langle\Psi|\hat{H}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = 4\lambda\left(\frac{3\hbar^2}{8m} + \frac{3m\omega^2}{32\lambda^2}\right) = \frac{3\hbar^2}{2m}\lambda + \frac{3m\omega^2}{8\lambda}.$$

$$\frac{\partial E_\lambda}{\partial\lambda} = \frac{3\hbar^2}{2m} - \frac{3m\omega^2}{8\lambda^2} = 0 \Rightarrow \lambda = \frac{m\omega}{2\hbar}$$

$$E_\lambda\left(\frac{m\omega}{2\hbar}\right) = \frac{3\hbar^2}{2\cancel{m}}\left(\frac{\cancel{m}\omega}{2\hbar}\right) + \frac{3m\omega^2}{8}\left(\frac{2\hbar}{m\omega}\right) = \frac{3}{4}\hbar\omega + \frac{3}{4}\hbar\omega$$

$$E_\lambda = \frac{3}{2}\hbar\omega.$$