Chem. 540
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Notes on Complex Numbers

A very useful representation of a complex number z is in terms of its modulus (absolute value) $|z|$ and phase ϕ. If

$$z = x + iy\,,$$

then

$$z = |z| e^{i\phi}$$

where $|z| = \sqrt{x^2 + y^2}$. This is the trigonometric form of the complex number, which is based on Euler's relation

$$e^{i\alpha} = \cos\alpha + i\sin\alpha\,.$$

The simplest way to convert a complex number to its trigonometric form is to write

$$z = \sqrt{x^2 + y^2}\left(\cos\phi + i\sin\phi\right)$$

and equate real and imaginary parts, obtaining

$$\cos\phi = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin\phi = \frac{y}{\sqrt{x^2 + y^2}}$$

From this we see that the phase is an angle whose tangent is y/x, i.e., we have $\tan\phi = y/x$. (Note, however, that there are two distinct points on the trigonometric circle with the same value of the tangent function, so we'll need to be sure we pick the correct one!) Since the values of the trigonometric functions remain unchanged upon adding a multiple of 2π to the angle, we can add $2n\pi$ (where n is an integer) to that angle.

Obviously, for any real number $\phi = 2n\pi$.

The trigonometric representation facilitates many calculations. For example, multiplication of two complex numbers is straightforward, as the phase of the product is equal to the sum of phases. Similarly, powers become very easy, e.g.,

$$z^k = (x + iy)^k = |z|^k e^{ik\phi}\,.$$

Inverses and roots are easily calculated as negative or fractional powers. For instance, to find the third (cube) roots of $z = 1 = e^{2n\pi i}$ one writes

$$z^{\frac{1}{3}} = e^{2n\pi i/3} = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\,.$$

Distinct roots occur for

$n=0: \quad z^{1/3} = 1$

$n=1: \quad z^{1/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i\,,$

$n=2: \quad z^{1/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i\,.$

(Other values of *n* produce the same results, so these are the only possibilities, consistent with the fundamental theorem of algebra.)